

On the Derivation and Interpretation of the Poincaré-Maxwell Group

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Abstract

The Lie algebra of the Poincaré-Maxwell group is derived in a manner that provides the interpretation of the equations of motion. It is clarified that the dynamics obtained from the orbit method is exactly equivalent to the classical description of the charged particle moving in the constant electromagnetic field. The multiplicity of the coadjoint orbits of the group under consideration is discussed.

1 Introduction

In the past years much attention has been attracted by $(2 + 1)$ dimensional models displaying Galilean symmetry that include non-commutative geometry [1–4]. In many works devoted to this subject [5–9] the Galilei group was studied both in free case and with external fields, also its central extensions have been investigated [10]. Other works devoted to the description of a charged particle moving in the electromagnetic field include [11–13]. There has also been some interest in similar considerations in the realm of relativistic theory [11, 14, 15] to which subject this paper aspires to contribute to. Similar techniques has been used in [16–23] to investigate models with Newton-Hooke like symmetries in various contexts.

The present work is greatly motivated by [14] and inspired by [24]. Its main purpose is to provide the interpretation of the equations of motion previously obtained in [14] therefore it is a straight continuation of the work therein. Recently, it was highlighted [25] that within the framework of the orbit method a physical interpretation is carried by a specific realisation of a Lie algebra via smooth functions rather than by an abstract Lie

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algebra. Towards this goal, a way of deriving the Poincaré-Maxwell group sparked by [24] is devised. This result displays the link between the approach consisting in building the dynamics on the coadjoint orbit of a proper Lie group with the more conventional framework of classical mechanics. This clarification on the meaning of the equations of motion obtained by the orbit method clears the way towards further investigations.

This paper is structured as follows. Section 2 provides a derivation of the Poincaré-Maxwell group in $(2 + 1)$ dimensions ($PM(2 + 1)$) carried out in a manner which lights up the road towards the interpretation of the equations of motion. Section 3 focusses on the different kinds of coadjoint orbits of $PM(2 + 1)$. Finally, in Section 4 a symplectic structure on the coadjoint orbit is given and the equations of motion are derived. Also, results of Section 2 are utilised to produce a clear interpretation of the dynamics. The article closes with some conclusions and comments on the relation with similar works in Section 5 where, also some future outlooks are discussed.

2 The Poincaré-Maxwell group

This section is focused on a relativistic particle endowed with a charge q moving in the xy plane under the influence of a constant electric field $\vec{E} = (E_x, E_y, 0)$ and a constant magnetic field perpendicular to the xy plane $\vec{B} = (0, 0, B)$. The Lie algebra of the Poincaré-Maxwell ($PM(2+1)$) group will be derived.

To focus the attention choose the symmetric gauge for the vector potential $\vec{A} = \frac{B}{2}[-y, x, 0]$ so that $\vec{B} = [0, 0, B]$ and the scalar potential $\phi = -(E_x x + E_y y)$ yielding $\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} = [E_x, E_y, 0]$. The Lagrangian for this system can be written as

$$\mathcal{L} = -m_0 c^2 \sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2}} - \frac{qBy\dot{x}}{2} + \frac{qBx\dot{y}}{2} + qE_x x + qE_y y \quad (2.1)$$

which yields the following generalised momenta

$$P_x = \frac{m_0 \dot{x}}{\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2}}} - \frac{qBy}{2}, \quad P_y = \frac{m_0 \dot{y}}{\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2}}} + \frac{qBx}{2} \quad (2.2)$$

and the Hamiltonian $\mathcal{H} = \vec{P} \cdot \dot{\vec{r}} - \mathcal{L}$ reads

$$\mathcal{H} = c \sqrt{\left(P_x + \frac{qBy}{2}\right)^2 + \left(P_y - \frac{qBx}{2}\right)^2} + m_0^2 c^2 - qE_x x - qE_y y. \quad (2.3)$$

Note that the coordinates (x, y, P_x, P_y) are canonical by construction. Moreover, the fields E_x , E_y and B will be treated as additional coordinates (keeping in mind that they are constant). In order to fit them into the Hamiltonian framework, one has to introduce their canonically conjugated momenta denoted π_x , π_y and β respectively. That is to assume

the Poisson bracket in the following form

$$\begin{aligned}\{F, G\} = & \left(\frac{\partial F}{\partial x} \frac{\partial G}{\partial P_x} - \frac{\partial F}{\partial P_x} \frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial G}{\partial P_y} - \frac{\partial F}{\partial P_y} \frac{\partial G}{\partial y} \right) \\ & + \left(\frac{\partial F}{\partial E_x} \frac{\partial G}{\partial \pi_x} - \frac{\partial F}{\partial \pi_x} \frac{\partial G}{\partial E_x} + \frac{\partial F}{\partial E_y} \frac{\partial G}{\partial \pi_y} - \frac{\partial F}{\partial \pi_y} \frac{\partial G}{\partial E_y} \right) \\ & + \left(\frac{\partial F}{\partial B} \frac{\partial G}{\partial \beta} - \frac{\partial F}{\partial \beta} \frac{\partial G}{\partial B} \right).\end{aligned}\quad (2.4)$$

Note that (2.4) implies that units of π_i and β are $[\pi_i] = [\text{A m s}^2]$ and $[\beta] = [\text{A m}^2 \text{s}]$. Next step is to find the integrals of motion i.e. the solutions the following differential equation

$$\{f, \mathcal{H}\} = 0 \quad (2.5)$$

where $f = f(x, y, P_x, P_y, E_x, E_y, \pi_x, \pi_y, B, \beta)$. One obvious solution is \mathcal{H} furthermore, since \mathcal{H} does not depend on π_x, π_y nor β there are solutions proportional to E_x, E_y and B . Take them to be $\mathcal{B} = qB, \mathcal{E}_x = -qE_x$ and $\mathcal{E}_y = -qE_y$. One more solution to (2.5) is $\mathcal{J} = xP_y - yP_x + E_x\pi_y - E_y\pi_x$.

The functions $\mathcal{H}, \mathcal{J}, \mathcal{E}_x, \mathcal{E}_y$ and \mathcal{B} together with a Poisson bracket (2.4) form a Lie algebra of a symmetry group of this system. Next, let us cast about the constants of motion i.e. the functions $f = f(t, x, y, P_x, P_y, E_x, E_y, \pi_x, \pi_y, B, \beta)$ such that

$$\{f, \mathcal{H}\} + \frac{\partial f}{\partial t} = 0. \quad (2.6)$$

The first two solutions to (2.6) have the dimension $\left[\frac{\text{kg m}}{\text{s}}\right]$, just as momentum does and they are of the form

$$\begin{aligned}\mathcal{P}_x &= P_x - \frac{qBy}{2} - qE_x t, \\ \mathcal{P}_y &= P_y + \frac{qBx}{2} - qE_y t,\end{aligned}\quad (2.7)$$

another two solutions have dimension $[\text{Jm}]$ and they read

$$\begin{aligned}\mathcal{K}_x &= xc \sqrt{\left(P_x + \frac{qBy}{2}\right)^2 + \left(P_y - \frac{qBx}{2}\right)^2 + m^2 c^2} \\ &\quad - c^2 t \left(P_x - \frac{qBy}{2} + \frac{qE_x t}{2}\right) - \frac{qE_y xy}{2} - \frac{qE_x x^2}{2} + c^2 B \pi_y - E_y \beta, \\ \mathcal{K}_y &= yc \sqrt{\left(P_x + \frac{qBy}{2}\right)^2 + \left(P_y - \frac{qBx}{2}\right)^2 + m^2 c^2} \\ &\quad - c^2 t \left(P_y + \frac{qBx}{2} - \frac{qE_y t}{2}\right) - \frac{qE_x xy}{2} - \frac{qE_y y^2}{2} + c^2 B \pi_x + E_x \beta.\end{aligned}\quad (2.8)$$

Consider a set of functions $(\mathcal{B}, \mathcal{E}_x, \mathcal{E}_y, \mathcal{H}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{K}_x, \mathcal{K}_y, \mathcal{J})$ (they shall be kept in this order). After calculating the Poisson bracket (2.4) for all the pairs chosen from the

above-mentioned set

$$\begin{array}{c}
\mathcal{B} \quad \mathcal{E}_x \quad \mathcal{E}_y \quad \mathcal{H} \quad \mathcal{P}_x \quad \mathcal{P}_y \quad \mathcal{K}_x \quad \mathcal{K}_y \quad \mathcal{J} \\
\left(\begin{array}{c}
\mathcal{B} \quad \{ \mathcal{B}, \mathcal{B} \} = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -\mathcal{E}_y \quad \mathcal{E}_x \quad 0 \\
\mathcal{E}_x \quad \{ \mathcal{B}, \mathcal{E}_x \} = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad c^2 \mathcal{B} \quad \mathcal{E}_y \\
\mathcal{E}_y \quad \{ \mathcal{B}, \mathcal{E}_y \} = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -c^2 \mathcal{B} \quad 0 \quad -\mathcal{E}_x \\
\mathcal{H} \quad \{ \mathcal{B}, \mathcal{H} \} = 0 \quad 0 \quad 0 \quad 0 \quad -\mathcal{E}_x \quad -\mathcal{E}_y \quad c^2 \mathcal{P}_x \quad c^2 \mathcal{P}_y \quad 0 \\
\mathcal{P}_x \quad \{ \mathcal{B}, \mathcal{P}_x \} = 0 \quad 0 \quad 0 \quad \mathcal{E}_x \quad 0 \quad \mathcal{B} \quad \mathcal{H} \quad 0 \quad \mathcal{P}_y \\
\mathcal{P}_y \quad \{ \mathcal{B}, \mathcal{P}_y \} = 0 \quad 0 \quad 0 \quad \mathcal{E}_y \quad -\mathcal{B} \quad 0 \quad 0 \quad \mathcal{H} \quad -\mathcal{P}_x \\
\mathcal{K}_x \quad \{ \mathcal{B}, \mathcal{K}_x \} = \mathcal{E}_y \quad 0 \quad c^2 \mathcal{B} \quad -c^2 \mathcal{P}_x \quad -\mathcal{H} \quad 0 \quad 0 \quad c^2 \mathcal{J} \quad \mathcal{K}_y \\
\mathcal{K}_y \quad \{ \mathcal{B}, \mathcal{K}_y \} = -\mathcal{E}_x \quad -c^2 \mathcal{B} \quad 0 \quad -c^2 \mathcal{P}_y \quad 0 \quad -\mathcal{H} \quad -c^2 \mathcal{J} \quad 0 \quad -\mathcal{K}_x \\
\mathcal{J} \quad \{ \mathcal{B}, \mathcal{J} \} = 0 \quad -\mathcal{E}_y \quad \mathcal{E}_x \quad 0 \quad -\mathcal{P}_y \quad \mathcal{P}_x \quad -\mathcal{K}_y \quad \mathcal{K}_x \quad 0
\end{array} \right)
\end{array}$$

one quickly notices that they form a family of Poisson algebras parameterised by t . Moreover, the commutation relations are identical for any value of t therefore all those algebras are mutually homomorphic. Therefore, putting $t = 0$ to pick a single algebra from that family results in no loss of generality. At the abstract level there is a 9 dimensional Lie algebra $\mathcal{PM}(2+1)$ (a Lie algebra of the Poincaré-Maxwell group) with the generators ($J_1 = B, J_2 = E_x, J_3 = E_y, J_4 = H, J_5 = P_x, J_6 = P_y, J_7 = K_x, J_8 = K_y, J_9 = J$) characterised by the following, non-vanishing, brackets

$$\begin{aligned}
[H, K_i] &= -c^2 P_j, & [B, K_i] &= \varepsilon_{ij} E_j, \\
[P_i, K_j] &= -\delta_{ij} H, & [H, P_i] &= E_i, \\
[K_i, K_j] &= -c^2 \varepsilon_{ij} J, & [E_i, K_j] &= -c^2 \varepsilon_{ij} B, \\
[P_i, J] &= -\varepsilon_{ij} P_j, & [P_i, P_j] &= -\varepsilon_{ij} B, \\
[K_i, J] &= -\varepsilon_{ij} K_j, & [E_i, J] &= -\varepsilon_{ij} E_j
\end{aligned} \tag{2.9}$$

with $i, j = x, y$.

Note that an algebra of this kind (with $c = 1$) was considered in [14] where, most significantly, the coadjoint action of its Lie group was found.

3 Coadjoint orbits

In what follows the coadjoint action of the $\mathcal{PM}(2+1)$ group will be presented recapping the results of [14] by applying the orbit method [26–28]. First of all, matrices of the adjoint action $m_{J_i}^{ad}$ corresponding to the generators J_1, \dots, J_9 are given by

$$m_{J_i}^{ad} = \begin{bmatrix} c_{i1}^1 & \cdots & c_{in}^1 \\ \vdots & \ddots & \vdots \\ c_{i1}^n & \cdots & c_{in}^n \end{bmatrix}$$

where c_{ij}^k are the structure constants readily available from (2.9). Let us denote by $(\mathcal{B}, \mathcal{E}_x, \mathcal{E}_y, \mathcal{H}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{K}_x, \mathcal{K}_y, \mathcal{J})$ a generic point of the space $\mathcal{PM}^*(2+1)$ in the basis dual to the basis $(B, E_x, E_y, H, P_x, P_y, K_x, K_y, J)$ of $\mathcal{PM}(2+1)$. Using the constants of motion found in the previous section as the coordinates will bring about the interpretation of the

equations of motion. The matrix of the coadjoint action corresponding to the group element

$$g = e^{bB} e^{d_x E_x} e^{d_y E_y} e^{\tau H} e^{a_x P_x} e^{a_y P_y} e^{n_x K_x} e^{n_y K_y} e^{\phi J}$$

is given by

$$M_g^{coAd} = e^{-\phi m_J^{ad}} e^{-n_y m_{K_y}^{ad}} e^{-n_x m_{K_x}^{ad}} e^{-a_y m_{P_y}^{ad}} e^{-a_x m_{P_x}^{ad}} e^{-\tau m_H^{ad}} e^{-d_y m_{E_y}^{ad}} e^{-d_x m_{E_x}^{ad}} e^{-b m_B^{ad}}.$$

The matrix exponentials of $m_{J_i}^{ad}$, $i = 1, \dots, 9$ are presented in the Appendix A. For the explicit form of the coadjoint action of the Poincaré-Maxwell group consult [14]. Having established the coadjoint action of the group $PM(2+1)$ it is time to identify its orbits. To this end consider the invariants of the coadjoint action, which are solutions to the following set of differential equations see [29–33] (note that AS means that the matrix is anti-symmetric)

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{E}_y & -\mathcal{E}_x & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & -c^2 \mathcal{B} & -\mathcal{E}_y \\ & & 0 & 0 & 0 & 0 & c^2 \mathcal{B} & 0 & \mathcal{E}_x \\ & & & 0 & \mathcal{E}_x & \mathcal{E}_y & -c^2 \mathcal{P}_x & -c^2 \mathcal{P}_y & 0 \\ & & & & 0 & -\mathcal{B} & -\mathcal{H} & 0 & -\mathcal{P}_y \\ & AS & & & & 0 & 0 & -\mathcal{H} & \mathcal{P}_x \\ & & & & & & 0 & -c^2 \mathcal{J} & -\mathcal{K}_y \\ & & & & & & & 0 & \mathcal{K}_x \\ & & & & & & & & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \mathcal{B}} \\ \frac{\partial}{\partial \mathcal{E}_x} \\ \frac{\partial}{\partial \mathcal{E}_y} \\ \frac{\partial}{\partial \mathcal{H}} \\ \frac{\partial}{\partial \mathcal{P}_x} \\ \frac{\partial}{\partial \mathcal{P}_y} \\ \frac{\partial}{\partial \mathcal{K}_x} \\ \frac{\partial}{\partial \mathcal{K}_y} \\ \frac{\partial}{\partial \mathcal{J}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (3.1)$$

There are three solutions to (3.1), namely

$$C_0 = \vec{\mathcal{E}}^2 - c^2 \vec{\mathcal{B}}^2, \quad (3.2)$$

$$C_1 = \mathcal{H}^2 - c^2 \vec{\mathcal{P}}^2 - 2 \left(\vec{\mathcal{K}} \cdot \vec{\mathcal{E}} - c^2 \vec{\mathcal{B}} \cdot \vec{\mathcal{J}} \right), \quad (3.3)$$

$$C_2 = \mathcal{H} \vec{\mathcal{B}} + \vec{\mathcal{P}} \times \vec{\mathcal{E}}. \quad (3.4)$$

Note that this method of finding the invariants of the coadjoint action does not require the knowledge of the explicit form of this action. Let us consider a generic point $\xi = [\mathcal{B}, \mathcal{E}_x, \mathcal{E}_y, \mathcal{H}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{K}_x, \mathcal{K}_y, \mathcal{J}] \in \mathcal{PM}^*(2+1)$, then the map

$$C : \mathcal{PM}^*(2+1) \rightarrow \mathbb{R}^3 \\ \xi \mapsto (C_0(\xi), C_1(\xi), C_2(\xi)) \quad (3.5)$$

has constant and maximal rank, moreover C is, by construction, constant at each point $\xi' = \text{coAd}_g(\xi)$, $g \in PM(2+1)$ belonging to the coadjoint orbit through ξ . Quick conclusion is that the preimage (each of its compact component, actually) of the point in \mathbb{R}^3 is an orbit of the coadjoint action which will be denoted $\mathcal{O}^{C_0, C_1, C_2}$. Let us attempt to find a parametrisation (equivalently cover it with maps) for the orbit $\mathcal{O}^{C_0, C_1, C_2}$. First let us consider a transformation

$$(\mathcal{B}, \mathcal{E}_x, \mathcal{E}_y, \mathcal{H}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{K}_x, \mathcal{K}_y, \mathcal{J}) \rightarrow (C_0, C_1, C_2, \mathcal{E}_x, \mathcal{E}_y, \mathcal{P}_x, \mathcal{P}_y, \mathcal{K}_x, \mathcal{K}_y) \quad (3.6)$$

its Jacobian equals $4\mathcal{B}^3$, therefore as long as $\mathcal{B} \neq 0$ the orbit, by the virtue of the implicit function theorem, is locally a graph of some function of $\mathcal{E}_x, \mathcal{E}_y, \mathcal{P}_x, \mathcal{P}_y, \mathcal{K}_x$ and \mathcal{K}_y . The case $\mathcal{B} = 0$ will be commented upon promptly. Since C_0, C_1 and C_2 are constant on the orbit, (3.2), (3.3) and (3.4) can be used to express \mathcal{B}, \mathcal{H} and \mathcal{J} in terms of $\mathcal{E}_x, \mathcal{E}_y, \mathcal{P}_x, \mathcal{P}_y, \mathcal{K}_x$ and \mathcal{K}_y . Firstly,

$$\mathcal{B}(\vec{\mathcal{E}}) = \pm \frac{1}{c} \sqrt{\vec{\mathcal{E}}^2 - C_0} \quad (3.7)$$

which means that for a fixed value of C_0 the condition $\vec{\mathcal{E}}^2 \geq C_0$ must hold for \mathcal{B} ought to be real. Next, from (3.4) and (3.7)

$$\mathcal{H}(\vec{\mathcal{E}}, \vec{\mathcal{P}}) = \frac{\vec{\mathcal{E}} \times \vec{\mathcal{P}}}{\mathcal{B}(\vec{\mathcal{E}})} + \frac{C_2}{\mathcal{B}(\vec{\mathcal{E}})} \quad (3.8)$$

and finally, keeping in mind (3.7) and (3.8) from (3.3) one has

$$\mathcal{J}(\vec{\mathcal{E}}, \vec{\mathcal{P}}, \vec{\mathcal{K}}) = \frac{C_1 - \mathcal{H}^2(\vec{\mathcal{E}}, \vec{\mathcal{P}}) + c^2 \vec{\mathcal{P}}^2 + 2\vec{\mathcal{K}} \cdot \vec{\mathcal{E}}}{2c^2 \mathcal{B}(\vec{\mathcal{E}})}. \quad (3.9)$$

One cannot but notice the ambiguity of (3.7) that will result in the diversity of the orbits. When dealing with high dimensional objects the only imagination friendly tool one has at disposal consists in taking the cross sections. In that spirit, notice that (3.2) or equivalently (3.7) defines, in the space spanned by $(\mathcal{B}, \mathcal{E}_x, \mathcal{E}_x)$, a surface which for $C_0 < 0$ is hyperboloid of two sheets, for $C_0 = 0$ it is a conical surface and for $C_0 > 0$ a hyperboloid of one sheet. See Figure 1. It is time to examine all the possibilities one at a time.

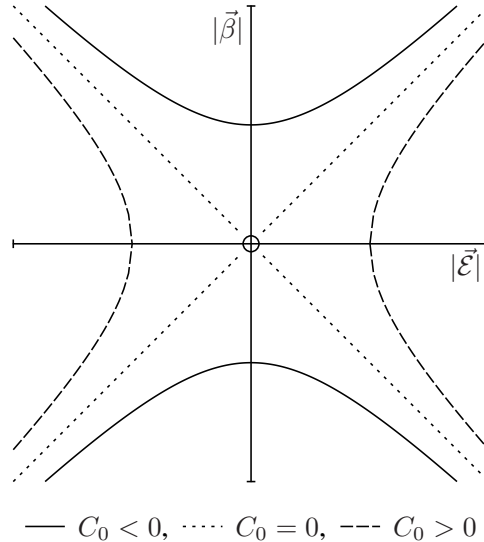


Figure 1: A sketch of the 3 dimensional cross section of the orbit in the space spanned by $(\mathcal{B}, \mathcal{E}_x, \mathcal{E}_x)$ depicting the cases when $C_0 < 0$, $C_0 = 0$ and $C_0 > 0$.

First, consider a fixed point $(C_0 < 0, C_1, C_2) \in \mathbb{R}^3$, its preimage by (3.5) consists of two coadjoint orbits in $\mathcal{PM}^*(2+1)$ denoted $\mathcal{O}_{\pm}^{C_0 < 0, C_1, C_2}$. Note that $\mathcal{B} \neq 0$ in this case.

Each of them can be covered by a single map and in terms of parametrisation they are defined by

$$\varphi : \left(\vec{\mathcal{E}}, \vec{\mathcal{P}}, \vec{\mathcal{K}} \right) \in \mathbb{R}^3 \mapsto \left(\mathcal{B}(\vec{\mathcal{E}}), \vec{\mathcal{E}}, \mathcal{H}(\vec{\mathcal{E}}, \vec{\mathcal{P}}), \vec{\mathcal{P}}, \vec{\mathcal{K}}, \mathcal{J}(\vec{\mathcal{E}}, \vec{\mathcal{P}}, \vec{\mathcal{K}}) \right) \in \mathcal{PM}^*(2+1)$$

with $\mathcal{B}(\vec{\mathcal{E}})$, $\mathcal{H}(\vec{\mathcal{E}}, \vec{\mathcal{P}})$ and $\mathcal{J}(\vec{\mathcal{E}}, \vec{\mathcal{P}}, \vec{\mathcal{K}})$ given by (3.7), (3.8) and (3.9) respectively where $C_0 < 0$.

Next, let us examine the preimage of $(C_0 = 0, C_1, C_2)$. As was said earlier, its three dimensional cross section obtained by fixing \mathcal{H} , $\vec{\mathcal{P}}$, $\vec{\mathcal{K}}$ and \mathcal{J} is a conical surface with a vertex at $(0, 0, 0)$. Not only does it contain points with $\mathcal{B} = 0$, it can not even be given the structure of a smooth manifold. The apparent riddle is easily unraveled by realising that if $\mathcal{B} = \mathcal{E}_x = \mathcal{E}_y = 0$ at any point of the coadjoint orbit it stays constant throughout that orbit. Therefore, this orbit (the cone is only its cross section) splits into three parts. The orbit $\mathcal{O}_{\mathcal{B}=0}^{C_0=0, C_1, C_2}$ containing the points for which $\mathcal{B} = \mathcal{E}_x = \mathcal{E}_y = 0$ is actually equivalent to the orbit of the Poincaré group and shall not be discussed in depth. Besides that, there are two orbits denoted $\mathcal{O}_{\pm}^{C_0=0, C_1, C_2}$ corresponding to the parts of the cone with positive and negative values of \mathcal{B} . Again, each of them can be covered by a single chart and is parametrised by

$$\varphi : \left(\vec{\mathcal{E}}, \vec{\mathcal{P}}, \vec{\mathcal{K}} \right) \in \mathbb{R}^3 \mapsto \left(\mathcal{B}(\vec{\mathcal{E}}), \vec{\mathcal{E}}, \mathcal{H}(\vec{\mathcal{E}}, \vec{\mathcal{P}}), \vec{\mathcal{P}}, \vec{\mathcal{K}}, \mathcal{J}(\vec{\mathcal{E}}, \vec{\mathcal{P}}, \vec{\mathcal{K}}) \right) \in \mathcal{PM}^*(2+1)$$

with $\mathcal{B}(\vec{\mathcal{E}})$, $\mathcal{H}(\vec{\mathcal{E}}, \vec{\mathcal{P}})$ and $\mathcal{J}(\vec{\mathcal{E}}, \vec{\mathcal{P}}, \vec{\mathcal{K}})$ given by (3.7), (3.8) and (3.9) respectively where $C_0 = 0$.

There is one more type of coadjoint orbits characterised by $C_0 > 0$. Its cross section of constant \mathcal{H} , $\vec{\mathcal{P}}$, $\vec{\mathcal{K}}$ and \mathcal{J} is a hyperboloid of one sheet. In principle it admits a single parametrisation by means of

$$(b, \phi) \mapsto \left(\mathcal{B} = b, \mathcal{E}_x = \sqrt{C_0 - c^2 \mathcal{B}^2} \cos \phi, \mathcal{E}_y = \sqrt{C_0 - c^2 \mathcal{B}^2} \sin \phi \right)$$

with $b \in \mathbb{R}$ and $0 \leq \phi < 2\pi$. However, it will be more convenient to assume $\mathcal{B} \neq 0$ discarding the case when particle moves in the constant electric field alone. Doing so, gives two parameterisations of the regions of the orbit disconnected by the plane $\mathcal{B} = 0$ which is sufficient for finding the equations of motion since the fields remain constant as the system evolves with time. Note that the plane $\mathcal{B} = 0$ corresponds to the motion of a charged particle in the constant electric field not considered here. Once again this two regions can be covered by a single chart and are parametrised by

$$\varphi : \left(\vec{\mathcal{E}}, \vec{\mathcal{P}}, \vec{\mathcal{K}} \right) \in \mathbb{R}^3 \mapsto \left(\mathcal{B}(\vec{\mathcal{E}}), \vec{\mathcal{E}}, \mathcal{H}(\vec{\mathcal{E}}, \vec{\mathcal{P}}), \vec{\mathcal{P}}, \vec{\mathcal{K}}, \mathcal{J}(\vec{\mathcal{E}}, \vec{\mathcal{P}}, \vec{\mathcal{K}}) \right) \in \mathcal{PM}^*(2+1)$$

with $\mathcal{B}(\vec{\mathcal{E}})$, $\mathcal{H}(\vec{\mathcal{E}}, \vec{\mathcal{P}})$ and $\mathcal{J}(\vec{\mathcal{E}}, \vec{\mathcal{P}}, \vec{\mathcal{K}})$ given by (3.7), (3.8) and (3.9) respectively where $C_0 > 0$ and $\vec{\mathcal{E}}^2 > C_0$. It is important to notice that such a 6 dimensional orbit has no immediate interpretation as a phase space of the system. The problem lies in the fact that besides the conjugate pair $\vec{\mathcal{K}}$ and $\vec{\mathcal{P}}$ the set of coordinates include $\vec{\mathcal{E}}$ which not only do not form a conjugate pair, they are rather artifactual since the fields are constant.

4 Symplectic structure and dynamics

The parametrisation φ can be used to derive the equations of motion (briefly recapitulating the results of [14] for the most part). Assuming that $\mathcal{B} \neq 0$ and fixing C_0 , C_1 and C_2 one finds that the Poisson structure on the coadjoint orbit, in the map corresponding to the parametrisation φ is given by

$$\Lambda^{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -c^2\mathcal{B} \\ 0 & 0 & 0 & 0 & c^2\mathcal{B} & 0 \\ 0 & 0 & 0 & -\mathcal{B} & -\mathcal{H} & 0 \\ 0 & 0 & \mathcal{B} & 0 & 0 & -\mathcal{H} \\ 0 & -c^2\mathcal{B} & \mathcal{H} & 0 & 0 & -c^2\mathcal{J} \\ c^2\mathcal{B} & 0 & 0 & \mathcal{H} & c^2\mathcal{J} & 0 \end{bmatrix} \quad (4.1)$$

where \mathcal{B} , \mathcal{H} and \mathcal{J} are given by (3.7), (3.8) and (3.9) respectively. The determinant of Λ equals $c^8\mathcal{B}^6$ which combined with the assumption $\mathcal{B} \neq 0$ means that the Poisson structure (4.1) is non-degenerate. Quickly inverting the matrix one finds the corresponding symplectic structure

$$\omega_{ij} = \begin{bmatrix} 0 & -\frac{\mathcal{J}}{c^2\mathcal{B}^2} - \frac{\mathcal{H}^2}{c^4\mathcal{B}^3} & \frac{\mathcal{H}}{c^2\mathcal{B}^2} & 0 & 0 & \frac{1}{c^2\mathcal{B}} \\ \frac{\mathcal{J}}{c^2\mathcal{B}^2} + \frac{\mathcal{H}^2}{c^4\mathcal{B}^3} & 0 & 0 & -\frac{\mathcal{H}}{c^2\mathcal{B}^2} & -\frac{1}{c^2\mathcal{B}} & 0 \\ -\frac{\mathcal{H}}{c^2\mathcal{B}^2} & 0 & 0 & \frac{1}{\mathcal{B}} & 0 & 0 \\ 0 & \frac{\mathcal{H}}{c^2\mathcal{B}^2} & -\frac{1}{\mathcal{B}} & 0 & 0 & 0 \\ 0 & \frac{1}{c^2\mathcal{B}} & 0 & 0 & 0 & 0 \\ -\frac{1}{c^2\mathcal{B}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (4.2)$$

Finally, the symplectic structure (4.2) together with the Hamiltonian (3.8) yields the equations of motion

$$\dot{\mathcal{E}}_i = 0, \quad \dot{\mathcal{P}}_i = -\mathcal{E}_i, \quad \dot{\mathcal{K}}_i = \mathcal{P}_i. \quad (4.3)$$

Combining (4.3) with (2.7) and (2.8) yields the familiar equation of motion

$$\frac{d}{dt} \left(\frac{m_0 \dot{\vec{r}}}{\sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}}} \right) = q \left(\dot{\vec{r}} \times \vec{B} + \vec{E} \right). \quad (4.4)$$

Note that the interpretation of (4.3), which does not contain the magnetic field, is problematic. However, equations (4.3) are equivalent to (4.4) thus the proper interpretation is upheld. Moreover, a clear link between the symplectic dynamics on the coadjoint orbits of the $PM(2+1)$ group and the conventional framework of classical mechanics is demonstrated.

5 Concluding remarks

Hereby a conclusion arises that the approach consisting in finding the dynamics on the coadjoint orbit of the Poincaré–Maxwell group is equivalent to the standard one. See the

discussion of the equations of motion on the coadjoint orbit of the Poincaré-Maxwell group presented in [14]. Although the mathematics involved in finding the coadjoint orbits of the group is significantly more complex than the standard Hamiltonian mechanics, one has to recognise that the Hamiltonian (3.8) is simpler than the initial one (2.3). That simplicity of the Hamiltonian (3.7) and the fact that the physical interpretation remains clear can be seen as an incentive for the further investigations. One of the possible paths to be taken goes down the road paved by [34] where a Stratonovich-Weyl quantisation was performed for the coadjoint orbits of the Galilei group in the free case (also see e.g. [35–37]).

A Appendix

In this Appendix the matrix exponentials of the matrices of the adjoint action $m_{J_i}^{ad}$ corresponding to the generators J_1, \dots, J_9 are presented.

$$\begin{aligned}
e^{bm_B^{ad}} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & e^{d_x m_{E_x}^{ad}} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -c^2 d_x & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -d_x \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
e^{d_y m_{E_y}^{ad}} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & c^2 d_y & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & d_y \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & e^{\tau m_H^{ad}} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \tau & 0 & -\frac{c^2 \tau^2}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \tau & 0 & -\frac{c^2 \tau^2}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -c^2 \tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -c^2 \tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

$$e^{a_x m_{P_x}^{ad}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -a_x & 0 & 0 & \frac{a_x^2}{2} \\ 0 & 1 & 0 & -a_x & 0 & 0 & \frac{a_x^2}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -a_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -a_x \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad e^{a_y m_{P_y}^{ad}} = \begin{bmatrix} 1 & 0 & 0 & 0 & a_y & 0 & 0 & 0 & \frac{a_y^2}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -a_y & 0 & 0 & 0 & \frac{a_y^2}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a_y & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_y \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{n_x m_{K_x}^{ad}} = \begin{bmatrix} \cosh cn_x & 0 & -c \sinh cn_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sinh cn_x}{c} & 0 & \cosh cn_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cosh cn_x & \frac{\sinh cn_x}{c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \sinh cn_x & \cosh cn_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cosh cn_x & -\frac{\sinh cn_x}{c} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c \sinh cn_x & \cosh cn_x \end{bmatrix}$$

$$e^{n_y m_{K_y}^{ad}} = \begin{bmatrix} \cosh cn_y & c \sinh cn_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sinh cn_y}{c} & \cosh cn_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cosh cn_y & 0 & \frac{\sinh cn_y}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \sinh cn_y & 0 & \cosh cn_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cosh cn_y & 0 & \frac{\sinh cn_y}{c} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c \sinh cn_y & 0 & \cosh cn_y \end{bmatrix}$$

$$e^{\phi m_J^{ad}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin \phi & \cos \phi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \phi & -\sin \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin \phi & \cos \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos \phi & -\sin \phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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